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Problems of Choosing Optimal Solutions for Systems with Random and Non-random Perturbations

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Abstract. The problem of choosing an optimal solution in stochastic optimization problem containing both random perturbations with given distributions and nonrandom perturbations about which only the regions of their possible values are known. As a criterion of optimality, the quantile criterion is used, i.e. the objective function value guaranteed with some given probability is optimised. This problem is closely connected with the problem of the construction confidence estimates for a statistically uncertain random vector that is a random vector with an incompletely known distribution. A concept of the generalized confidence set is used for statistically uncertain vector, and its properties are studied. The quantile stochastic optimization problem under incomplete information is solved by means of an optimal choice of the generalized confidence region.

INTRODUCTION

In many practical optimization problem a purpose function depends on random perturbations. As a rule parameters of the probability distributions are incompletely known and estimated on the statistical data. The modern approach to the stochastic optimization is the using of the quantile criterion. In the financial analysis it is called VaR – Value at Risk.

The quantile optimization problem is closely connected with the problem of the confidence estimation for a random vector. One of possible methods of the finding a solution is the choosing of an optimal confidence set of the purpose function [1]. On the other hand there are many confidence sets for a given random vector corresponding to the same probability level, so the confidence estimation problem may be reformulated as a stochastic optimization problem with a quantile criterion.

If the purpose function depends on random parameters with incompletely known distributions then the problem may be reduced to the optimal choice of a confidence estimate for these parameters. A random vector with an incompletely known distribution is called a statistically uncertain vector. In the paper [2] a concept of the generalized confidence set for a statistically uncertain vector was introduced, and properties of these sets were studied.

The quantile stochastic optimization problem under incomplete information may be solved by means of an optimal choice of the generalized confidence set [3]. In [4] the method of finding a suboptimal solution of the quantile optimization problem under uncertainty was suggested.

In the paper [3] the problem of the optimal choice of parameters of a runway were considered as a quantile optimization problem in condition that the wind velocity is modeled as a statistically uncertain vector. The problem of forecasting the state probabilities vector for stationary Markov chain in case of transition probabilities are not exactly known was investigated in [5] by using confidence sets for rows of the matrix of transition probabilities.

CONFIDENCE ESTIMATION OF STATISTICALLY UNCERTAIN VECTOR

Let (Ω, \mathcal{A}, P) be a probability space, $\mathcal{B}^{(n)}$ be σ -algebra of all Borel sets on \mathbb{R}^n .

Definition 1.[2] A map $\xi(\omega, Z) : \Omega \times Z \rightarrow \mathbb{R}^n$ is called a statistically uncertain random vector if:

1. the function $\xi(\omega, z)$ is a random vector for any fixed $z \in Z$, i.e. the set $\{\omega : \xi(\omega, z) \in B\} \in \mathcal{A}$ is measurable for any $B \in \mathcal{B}^{(n)}$, $z \in Z$;

2. the probability $P_z(B) = P\{\xi(\omega, z) \in B\}$ is a continuous function with respect to z for any fixed $B \in \mathcal{B}^{(n)}$;
3. the set Z is a compact set consisted of more than one point.

A generalized random vector $\tilde{\xi}(\omega, Z)$ is called continuous if for any $z \in Z$ the random vector $\xi(\omega, z)$ has an absolutely continuous distribution, i.e. it has a density function.

Definition 2. A set $\tilde{X}_\alpha \subset \mathcal{B}^{(n)}$ is called a generalized confidence set with level α for a statistically uncertain random vector $\tilde{\xi}(\omega, Z)$, if

$$\mathcal{P}\{\tilde{\xi}(\omega, Z) \in X_\alpha\} = \alpha,$$

where

$$\mathcal{P}\{\tilde{\xi}(\omega, Z) \in X_\alpha\} = \min_{z \in Z} P\{\xi(\omega, z) \in X_\alpha\}.$$

Generalized confidence sets (as well as standard confidence sets) are not uniquely defined: there are many generalized confidence regions corresponding to a fixed probability α .

Usually a union of confidence sets corresponding to all permissible distributions is taken as a confidence set in the statistically uncertain case. It was shown in [2, 3] that this estimator may be improved in the most cases by means of generalized confidence sets.

Theorem 1. Let $\tilde{\xi}(\omega, Z)$ be a statistically uncertain continuous random vector, $\{X_z^\alpha \mid z \in Z\}$ be a family of confidence sets with the level α for $\xi(\omega, z)$, i.e. for any $z \in Z$ the relation $P\{\xi(\omega, z) \in X_z^\alpha\} = \alpha$ holds.

If the union

$$\hat{X}_\alpha = \bigcup_{z \in Z} X_z^\alpha, \quad (1)$$

is a measurable set, then \hat{X}_α is a generalized confidence set with a level $\alpha_1 \geq \alpha$ for the statistically uncertain random vector $\tilde{\xi}(\omega, Z)$.

The equality $\alpha_1 = \alpha$ holds if and only if there exists such a parameter $z^* \in Z$ such that $P\{\xi(\omega, z^*) \in \hat{X}_\alpha\} = \alpha$.

Theorem 2. Let $\tilde{\xi}(\omega, Z) = \{z + \eta(\omega) \mid z \in Z\}$ be a statistically uncertain vector and the following conditions hold

1. $\eta(\omega)$ is a continuous random vector with a given density function $f_\eta(x)$ such that $f_\eta(x) > 0$ for all $x \in \mathbb{R}^n$;
2. $Z \subset \mathbb{R}^n$ is a given convex compact set;
3. B_α is a convex compact confidence set for $\eta(\omega)$ with a level $\alpha \in (0.5; 1)$ and $0 \in \text{int}(B)$, where $\text{int}(B)$ is the set of all interior points of B .

Then there exists $\varepsilon \in (0, 1)$ such that the set $Z + \varepsilon B_\alpha$ is a generalized confidence region of probability α for $\tilde{\xi}(\omega, Z)$.

In the paper [2] the generalized confidence sets were studied for the statistically uncertain vector $\tilde{\xi}(\omega, Z) = \{z + \eta \mid z \in Z\}$, where Z is a known set (a line segment, a ball or an ellipsoid) and η is the gaussian random vector with zero mean value and a nongenerate covariance matrix.

Let us consider a simple example of the construction of generalized confidence sets.

Example 1. Let us find generalized confidence sets for the statistically uncertain random value

$$\tilde{\xi}(\omega, Z) = \{z + \eta(\omega) \mid z \in Z\},$$

where Z is the interval $Z = [-a, a]$, $\eta(\omega)$ is the normally distributed random value $\eta \sim \mathcal{N}(0, \sigma)$.

The sets $X_\alpha(0) = [-t_{0.5\alpha}\sigma, t_{0.5\alpha}\sigma]$, are confidence sets for $\eta(\omega)$ with a probability α . Here

$$\Phi(t_\alpha) = \alpha, \quad \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_0^t \exp\left(-\frac{z^2}{2}\right) dz.$$

A symmetrical generalized confidence set \tilde{X}_α for $\tilde{\xi}(\omega, Z)$ has the form

$$\tilde{X}_\alpha = [-a - \Delta_\alpha; a + \Delta_\alpha], \quad \Delta_\alpha = \sigma \text{gal}(\alpha, \sigma^{-1}a)$$

where $g = \text{gal}(\alpha, a)$ is the root of the equation

$$\Phi(g + 2v) + \Phi(g) = \alpha. \quad (2)$$

Properties of the function $\text{gal}(\alpha, v)$

1. $\text{gal}(\alpha, v)$ is defined and continuous on $(0.5; 1) \times [0; +\infty)$;
2. $\text{gal}(\alpha, v)$ increases on α and decreases on v for $v \geq 0$;
3. $t_{\alpha-0.5} < \text{gal}(\alpha, v) < t_{\alpha/2}$ for any $v > 0$;
4. $\text{gal}(\alpha, 0) = t_{\alpha/2}$, $\lim_{v \rightarrow +\infty} \text{gal}(\alpha, v) = t_{\alpha-0.5}$;
5. $\text{gal}(\alpha, v_1 + v_2) \leq \text{gal}(\alpha, v_1) + \text{gal}(\alpha, v_2)$.

One can calculate the function $\text{gal}(\alpha, v)$ by using a standard mathematical program such as MathCad, MatLab, Mathematica, etc. On the other hand this implicitly given function has a simple approximation:

$$\text{gal}(\alpha, v) \approx g_1(\alpha, v) = t_{\alpha/2} - v + 0.5t_{\alpha/2}v^2 \quad \text{for } 0 \leq v < t_{\alpha/2}^{-1}.$$

This approximation may be used as an upper estimation for $\text{gal}(\alpha, v)$, since the following inequality holds:

$$\text{gal}(\alpha, v) \leq \hat{g}_1(v) = \begin{cases} g_1(v) = t_{\alpha/2} - v + 0.5t_{\alpha/2}v^2 & \text{if } 0 \leq v < t_{\alpha/2}^{-1} \\ g_1(t_{\alpha/2}^{-1}) = t_{\alpha/2} - 0.5(t_{\alpha/2})^{-1} & \text{if } v > t_{\alpha/2}^{-1}. \end{cases} \quad (3)$$

A more accurate approximation has the form

$$\text{gal}(\alpha, v) \approx g_2(\alpha, v) = t_{\alpha-0.5} + B_\alpha \exp(-C_\alpha v - D_\alpha v^2) \quad \text{for all } v \geq 0.$$

The function $g_2(\alpha, v)$ is the an upper estimate of $\text{gal}(\alpha, v)$ too, i.e.

$$\text{gal}(\alpha, v) \leq g_2(\alpha, v) = t_{\alpha-0.5} + B_\alpha \exp(-C_\alpha v - D_\alpha v^2) \quad \text{for all } v \geq 0. \quad (4)$$

In the Picture 1 and Picture 2 the function $\text{gal}(\alpha, v)$ (the blue line) and its upper approximations $\hat{g}_1(\alpha, v)$ (the green line with with square markers), $g_2(\alpha, v)$ (the red line with triangle markers) are presented for $\alpha = 0.9$. and $\alpha = 0.99$. The grey dotted line is drawn at the level of $t_{\alpha-0.5}$.

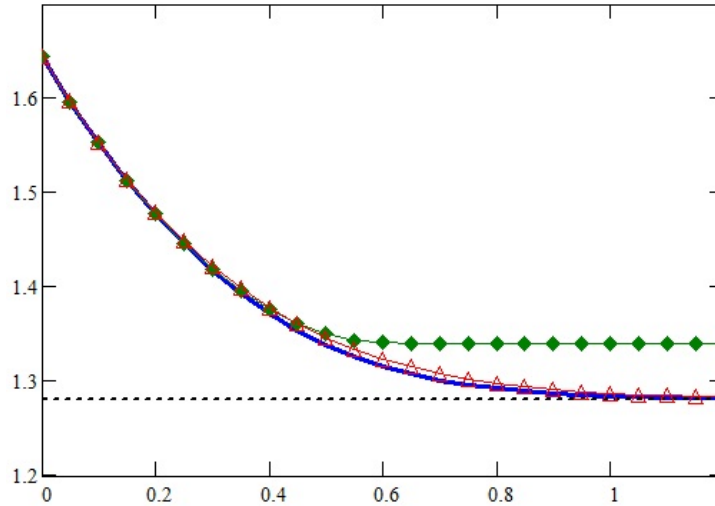


FIGURE 1. Function $\text{gal}(\alpha, v)$ and its approximations for $\alpha = 0.9$

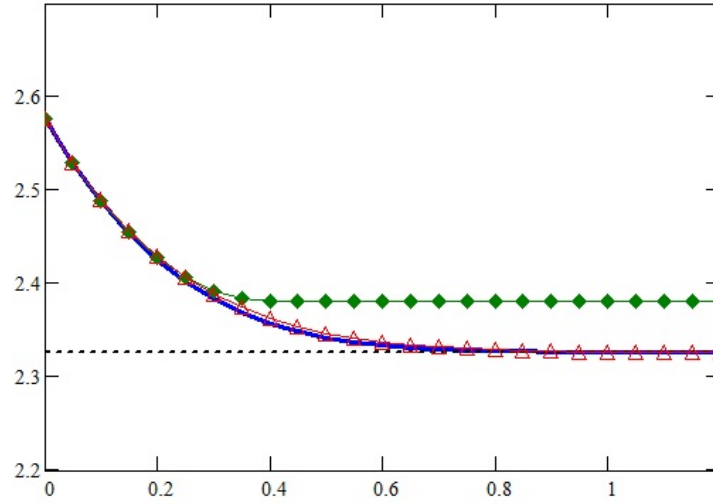


FIGURE 2. Function $\text{gal}(\alpha, v)$ and its approximations for $\alpha = 0.99$

QUANTILE OPTIMIZATION PROBLEM UNDER UNCERTAINTY

Let us consider a stochastic optimization problem with the quantile criterion in conditions of incomplete information on a probabilistic distribution. Suppose that the function $F(x, \xi(\omega, Z))$ is to be minimized by optimal choice of the vector $x \in X$, here $\xi(\omega, Z) = \{\xi(\omega, z) | z \in Z\}$ is a stochastically uncertain random vector. Denote a quantile of the function $F(x, \xi(\omega, Z))$ corresponding to a probability α by $q_\alpha(x)$,

$$q_\alpha(x) = \min\{q : \mathcal{P}\{F(x, \xi(\omega, Z)) \leq q\} \geq \alpha\} = \min\{q : \min_{z \in Z} P\{F(x, \xi(\omega, z)) \leq q\} \geq \alpha\}. \quad (5)$$

The optimization problem for the quantile has the form:

$$q_\alpha(x) \rightarrow \min_{x \in X}. \quad (6)$$

The optimal solution of the quantile optimization problem (5)–(6) may be reduced to the optimal choice of the generalized confidence set for the statistically uncertain vector $\xi(\omega, Z)$.

Theorem 4.[4] Let $\xi(\omega, Z)$ be a statistically uncertain continuous random vector, the function $F(x, y)$ be continuous and the minimum in (5) be reached then

$$\tilde{q}_\alpha^* = \min_{x \in X} \min_{\tilde{E}_\alpha \in \tilde{\mathcal{E}}_\alpha} \max_{y \in \tilde{E}_\alpha} F(x, y), \quad (7)$$

where $\tilde{\mathcal{E}}_\alpha \subset \mathcal{B}^{(n)}$ is the family of generalized confidence sets \tilde{E}_α with level not less than α :

$$\tilde{\mathcal{E}}_\alpha = \{\tilde{E}_\alpha \in \mathcal{B}^{(n)} : \mathcal{P}\{\xi(\omega, Z) \in \tilde{E}_\alpha\} \geq \alpha\}.$$

In most cases this problem is not easy solved. Let us consider the following example of the minimization of the quantile criterion in the statistically uncertain case.

Example 2. Let find the value x minimizing the distance

$$F(x, \xi(\omega, Z)) = |\xi(\omega, Z) - x| \rightarrow \min_x, \quad (8)$$

where statistically uncertain value $\xi(\omega, Z)$ is the product of a gaussian random value with known parameters $\eta(\omega) = \eta \sim N(m, \sigma)$, $m > 0$, and a nonrandom uncertain value $z \in Z$, Z is a given interval $Z = [b_1, b_2]$, $0 < b_1 < b_2$, i.e.

$$\xi(\omega, Z) = \{z \cdot \eta(\omega), z \in Z = [b_1, b_2]\}. \quad (9)$$

The quantile criterion for (8) has the form: for a fixed probability α to find a value x minimizing the quantile:

$$q_\alpha(x) \rightarrow \min_x, \quad (10)$$

$$q_\alpha(x) = \min\{q : P\{|z \cdot \eta - x| \leq q\} \geq \alpha \text{ for all } z \in [b_1, b_2]\}.$$

An equivalent problem is to find the minimal generalized confidence set \hat{X}_α of the level α for the random vector with incompletely known distribution $\xi(\omega, Z) = \{z \cdot \eta, z \in [b_1; b_2]\}$ and to take a center of this set as an optimal solution x in (10).

Let the probability α satisfies the condition $c_\alpha^- = m - t_{\alpha/2}\sigma > 0$. The set $X_\alpha(z) = [zc_\alpha^-; zc_\alpha^+]$ is a confidence set of the level α for $\xi(z, \eta) = z \cdot \eta$. Their union \hat{X}_α has the form

$$\hat{X}_\alpha = \bigcup_z X_\alpha(z) = [b_1 c_\alpha^-; b_2 c_\alpha^+].$$

Calculations show that the minimal probability

$$p_- = \mathcal{P}\{\xi(\omega, Z) \in \hat{X}_\alpha\} = \min_{z \in Z} P\{z \cdot \eta \in [b_1 c_\alpha^-; b_2 c_\alpha^+]\} > \alpha.$$

And in specific examples, this difference is greater than 0.05. Let take for example

$$b_1 = 2, b_2 = 4, m = 2, \sigma = 0.2, \alpha = 0.9.$$

The set $\hat{X}_{0.9} = [2.684; 7.974]$ with the center in $\hat{x} = 5.329$ and $\hat{q}(\hat{x}) = 2.645$. But the minimal probability

$$p_- = \min_{z \in [2, 4]} P\{z \cdot \eta \in [2.684; 7.974]\} = 0.95 > 0.9,$$

therefore the confidence estimate can be reduced. The calculation shows that the minimal quantile in problem (10) is equal to $q(x^*) = 2.302 < 2.645$ and the optimal value $x^* = 5.277$.

Conclusion

In the paper the quantile optimization problem is considered in the case of incomplete information about distributions of random parameters. The problem is solved by means of generalised confidence sets for statistically uncertain vector. These sets are studied in detail for the sum of uncertain parameter and gaussian random value, obtained results applied to find the optimal quantile.

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